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A symmetric nonlocal damage theory

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Abstract

The paper presents a thermodynamically consistent formulation for nonlocal damage models. Nonlocal models have been recognized as a theoretically clean and computationally efficient approach to overcome the shortcomings arising in continuum media with softening. The main features of the presented formulation are: (i) relations derived by the free energy potential fully complying with nonlocal thermodynamic principles; (ii) nonlocal integral operator which is self-adjoint at every point of the solid, including zones near to the solid's boundary; (iii) capacity of regularizing the softening ill-posed continuum problem, restoring a meaningful nonlocal boundary value problem. In the present approach the nonlocal integral operator is applied consistently to the damage variable and to its thermodynamic conjugate force, i.e. nonlocality is restricted to internal variables only. The present model, when associative nonlocal damage flow rules are assumed, allows the derivation of the continuum tangent moduli tensor and the consistent tangent stiffness matrix which are symmetric. The formulation has been compared with other available nonlocal damage theories.

Finally, the theory has been implemented in a finite element program and the numerical results obtained for 1-D and 2-D problems show its capability to reproduce in every circumstance a physical meaningful solution and fully mesh independent results.

Keywords: Nonlocal damage; Thermodynamics; Localization limiter; Associative damage

1. Introduction

Post-elastic behaviour of quasi-brittle materials is micro-mechanically characterized by many complex mechanisms, such as nucleation, coalescence and development of micro-cracks and micro-defects. These complex dissipative mechanisms, to some extent, can be phenomenologically represented by continuum damage constitutive models (Lemaitre and Chaboche, 1990; Krajcinovic, 1996). Besides the degradation of the material elastic properties, damage induces a remarkable overall strength reduction when a certain damage threshold is attained. This phenomenon, called strain softening, has been recognized as a source of theoretical and computational difficulties for continuum based structural modelling, which are substantially

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originated by constitutive instability matters. Classical softening continuum descriptions display the localization of the strains into a band of zero thickness with the paradoxal consequence of structural failure with zero energy dissipation. On the other hand, standard finite element discrete solutions are meaningless, being their results pathologically affected by the employed mesh size and orientation (damage tends to concentrate only along one single finite element layer). In order to overcome the above mentioned shortcomings many remedies have been proposed, which are commonly called regularization techniques.

The most simple regularization technique is probably the fracture energy regularization approach (Bažant and Cedolin, 1979; Bažant and Oh, 1983; Este and Willam, 1992; Comi and Perego, 2001a). This technique is basically a computational strategy for finite element solutions which, observing that damage localizes into a band one element thick, defines a material fracture energy density scaled according to the size of the elements. In this way the energy dissipated into the band is an objective quantity and the overall structural response is mesh independent, at least as long as the element size is kept greater of the localization band. Despite the fact that this approach is quite simple and also efficient to apply, it does not provide the spatial damage distribution into the band, suffers of the lack of convergence for mesh size smaller and smaller inside the band thickness and it has neither a continuum counterpart nor a micromechanical justification.

Advanced regularization techniques require the definition of enriched continuum models, in which long range interaction forces, traditionally neglected in classical continuum theories, appear. These theories have as common feature the presence of one, or more, internal length parameters. The internal length has to be intended as a material constitutive parameter aimed to describe the nonlocal micro-interactions typically produced in an heterogeneous media suffering a micro-decohesion (i.e. damage) process.

Among these theories the most robust regularization approaches, with clear micro-mechanical interpretations, are the so-called nonlocal approaches. Nonlocal approaches are defined either in a strong form (spatial integral) (Eringen, 1981; Pijaudier-Cabot and Bažant, 1987; Borino et al., 1999; Ganghoffer et al., 1999) or in a weak form (spatial higher gradients) (Aifantis, 1984, 1992; de Borst and Mühlhaus, 1992; de Borst and Pamin, 1996; Addessi et al., 2002). Substantially, for nonlocal materials the constitutive relations are not pointwise relations but rather they involve either integral average values, or higher spatial gradients. In every nonlocal model the material reproduced is not a "simple material", in the sense of Noll (1972). Nonlocal theories where originally developed for the description of linear elastic heterogeneous materials and often applied to linear elastic fracture mechanics problems where high stress gradient fields are observed in the neighbour of sharp cracks with stress singularities at the crack tips. Nonlocal elastic theories allow to remove the stress singularity and to recover a regularized finite value stress field everywhere. Recently, nonlocal elastic models are returned to be the subject of new studies and ongoing researches (Polizzotto, 2001), also because they can be closely related to heterogeneous continua and continua with micro-structure (Luciano and Willis, 2001).

Nonlocal models have been introduced for strain softening materials with a rather different intent from the one pursued in elasticity. Namely in the softening regimes, no matter if induced by damage or by strain-softening plasticity, the purpose is to regularize the inelastic dissipative modes associated to the strain softening and then to re-establish a well posed boundary value problem. In this view the variables that need to be regularized are solely the ones related to the dissipative mechanism, i.e. the ones that explicitly appear in the dissipation functional. As observed by Pijaudier-Cabot and Bažant (1987), in the specific damage context, the damage variable in itself is a perfect candidate for being treated as nonlocal. This choice has a further strength since physical justifications on the ground of micro-crack interactions, stored energy release from a finite zone around the crack and material heterogeneity, can be invoked (Bažant, 1984, 1991; Bažant and Planas, 1998).

In this paper a recent approach to nonlocal dissipative problems is extended to nonlocal damage problems. The theoretical framework is based on a thermodynamic formulation originally developed in the context of strain-softening plasticity (Polizzotto et al., 1998; Borino et al., 1999). At difference of Benvenuti et al. (2000, 2002), the present approach considers as nonlocal the damage variable rather than the damage-hardening internal variable and then, as observed by Pijaudier-Cabot and Bažant (1987), Comi (2001) and Comi and Perego (2001b,c), it allows, in case of linear isotropic damage models, a rather effective numerical implementation. In fact, for this choice of the nonlocal variable in the predictor-corrector classical iterative computational scheme, the damage corrector phase can be enforced locally at each Gauss integration point without the honerous nonlocal iterative correction scheme used in Strömberg and Ristimnaa (1996), Borino and Failla (2000), and Benvenuti et al. (2000, 2002). Besides being thermodynamically consistent, the present nonlocal damage formulation utilizes a nonlocal integral operator which is self-adjoint, ensuring the preservation of uniform fields for both the nonlocal damage and its thermodynamic conjugate force at every point of the domain, including the zones close to the body's boundary avoiding the difficulties observed by Comi and Perego (2001c).

The paper is organized in the following fashion. After the present Introduction, the nonlocal damage concepts are presented in Section 2 and the new idea of symmetric nonlocal weight operator is discussed and justified. Section 3 is devoted to framing the model in general thermodynamics of nonlocal media. Section 4 presents the associative damage laws and shows how to derive a principle of maximum dissipation consistent with this nonlocal context. In order to show that the present nonlocal approach can effectively regularize the problem, a localization analysis based on the dispersive analysis of the propagation of stress waves is reported in Appendix A. Section 5 is devoted to the study of the material response to an assigned total strain rate field. A material variational principle is presented and the nonlocal continuum tangent operator is derived, showing its symmetry. In Section 6 the structural rate problem is examined and a kinematical variational principle proposed. Section 7 describes the finite element implementation of the model, discusses the iterative incremental procedure of Newton–Raphson and some specific aspects related to the present nonlocal context. Section 8 is aimed to show numerical results obtained with the present formulation. The first example is concerned with a classical 1-D bar in uniform traction whereas the second example analyzes a plane stress indented plate subjected to traction. The results obtained are physical meaningful and fully mesh independent. Finally some conclusive comments are reported in Section 9.

2. Nonlocal damage model

Let us consider a body that in its undeformed state occupy the domain V, of the three dimensional Euclidean space, with boundary ∂V. We confine our formulation to the case of small induced strains and admit that the straining process may promote linear isotropic damage so that the stress–strain relation is given by

$$\boldsymbol{\sigma} = (1 - \bar{d})\boldsymbol{D}^{\mathrm{e}} : \boldsymbol{\varepsilon}, \tag{1}$$

where $\boldsymbol{D}^{\mathrm{e}}$ is the fourth order elastic moduli tensor, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the stress and the strain tensor, respectively, and finally $\bar{d}$ is the nonlocal isotropic damage variable. Eq. (1) is a nonlocal constitutive relation since $\bar{d}(\boldsymbol{x})$ is obtained by a spatial weight average applied to a local damage variable, $d(\boldsymbol{x})$, by means of the following integral relation

$$\bar{d}(\boldsymbol{x}) = \int_V W(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}(\boldsymbol{y})\, \mathrm{d}V(\boldsymbol{y}). \tag{2}$$

In Eq. (2) $W(\boldsymbol{x}, \boldsymbol{y})$ is a space weight function which describes the mutual nonlocal damage interactions. For physically sound formulations, the weight function $W(\boldsymbol{x}, \boldsymbol{y})$ is required to be positive, to have its maximum value for $r = \|\boldsymbol{x} - \boldsymbol{y}\| = 0$ (i.e. for $\boldsymbol{x} = \boldsymbol{y}$) and to decrease monotonically and rapidly to zero for increasing r (i.e. the nonlocal interactions are effective only in a small, but finite, neighbour of the field point).

Moreover, since it is expected that uniform damage fields do not suffer alterations by the spatial average procedure of Eq. (2) the following normalization condition is required

$$\int_V W(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}V(\boldsymbol{y}) = 1 \quad \forall \boldsymbol{x} \in V. \tag{3}$$

In order to impose such a condition at every point of the body, including the points close to the body's boundary, where the effective contribution space domain is reduced, and then the integral of Eq. (3) is penalized, Pijaudier-Cabot and Bažant (1987) (see also Strömberg and Ristimnaa (1996) for nonlocal plasticity) proposed a rather nonstandard weighting approach, posing

$$W(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\Omega_{\mathrm{r}}(\boldsymbol{x})}\alpha(\boldsymbol{x},\boldsymbol{y}), \tag{4}$$

where

$$\Omega_{\mathrm{r}}(\boldsymbol{x}) = \int_V \alpha(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}V(\boldsymbol{y}). \tag{5}$$

$\alpha(\boldsymbol{x},\boldsymbol{y})$ is an attenuation function which depends only on the distance r and thus enjoys the property $\alpha(\boldsymbol{x},\boldsymbol{y}) = \alpha(\boldsymbol{y},\boldsymbol{x})$. $\Omega_{\mathrm{r}}(\boldsymbol{x})$, defined in Eq. (5), is called representative volume.

Typical choices for the attenuation function $\alpha(\boldsymbol{x},\boldsymbol{y})$ are:

$$\alpha(\boldsymbol{x},\boldsymbol{y}) = k_{\mathrm{G}} \exp(||\boldsymbol{x}-\boldsymbol{y}||^2/\ell_0^2) \tag{6}$$

or

$$\alpha(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} k_{\mathrm{B}}(1-||\boldsymbol{x}-\boldsymbol{y}||^2/R^2)^2 & \text{if } ||\boldsymbol{x}-\boldsymbol{y}|| \leqslant R, \\ 0 & \text{if } ||\boldsymbol{x}-\boldsymbol{y}|| > R. \end{cases} \tag{7}$$

The weight function of Eq. (6), which is the Gauss function, has an unbounded support, whereas Eq. (7), which is denoted bell function, is defined for $r = ||\boldsymbol{x}-\boldsymbol{y}|| \leqslant R$. In both cases ℓ_0 and R play the role of an internal length parameter which controls the nonlocal spatial spread of the damage. It can be observed that the representative volume $\Omega_{\mathrm{r}}(\boldsymbol{x})$, when the point $\boldsymbol{x}$ is far from the body's boundary, tends to become a constant value, that can be denoted Ω_∞, since it is exactly the value that would be obtained if the body were unbounded, i.e. extended as the entire Euclidean three dimensional space.

It is to remark that the weight function of Eq. (4) is not symmetric, i.e. $W(\boldsymbol{x},\boldsymbol{y}) \neq W(\boldsymbol{y},\boldsymbol{x})$, and this is due to the necessity to accommodate the condition of uniform field near to the boundaries. The fact that the kernel of the integral Eq. (2) is not symmetric gave not so many problems in the original formulation, Pijaudier-Cabot and Bažant (1987), besides the accepted fact of the nonsymmetry of the tangent operators. However, when Comi and Perego (2001b) tried to apply recent concepts of nonlocal thermodynamics, (Polizzotto et al., 1998; Borino et al., 1999), they found unexpected problems which gave the impression that a symmetric nonlocal damage formulation was theoretically appealing but not very suitable for practical computational purposes.

In the present paper we introduce a new approach to the nonlocal integral averaging weight function based on the following assumption:

$$W(\boldsymbol{x},\boldsymbol{y}) = \left(1 - \frac{\Omega_{\mathrm{r}}(\boldsymbol{x})}{\Omega_\infty}\right)\delta(\boldsymbol{x},\boldsymbol{y}) + \frac{1}{\Omega_\infty}\alpha(\boldsymbol{x},\boldsymbol{y}), \tag{8}$$

where $\delta(\boldsymbol{x},\boldsymbol{y})$ is the Dirac delta function and $\alpha(\boldsymbol{x},\boldsymbol{y})$ is the same attenuation function defined in Eq. (6) or (7). Inserting Eq. (8) into the nonlocal definition given by Eq. (2) we obtain

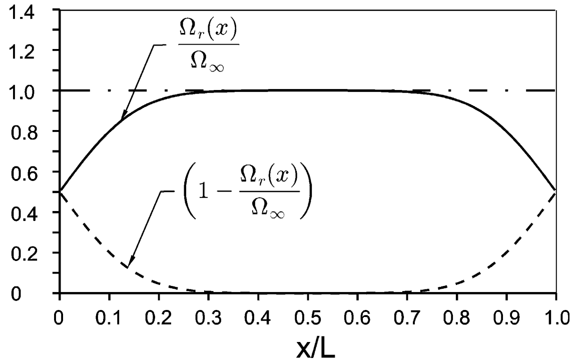

Fig. 1. Diagram showing the contribution of the first local term $(1-\Omega_r(x)/\Omega_\infty)$ and of the second term $\Omega_r(x)/\Omega_\infty$ for one-dimensional bar in uniform state of damage. Results obtained using a Gauss attenuation function with internal length $\ell_0 = 0.3L$.

$$\bar{d}(\boldsymbol{x}) = \left(1 - \frac{\Omega_r(\boldsymbol{x})}{\Omega_\infty}\right) d(\boldsymbol{x}) + \frac{1}{\Omega_\infty}\int_V \alpha(\boldsymbol{x},\boldsymbol{y}) d(\boldsymbol{y})\,\mathrm{d}V(\boldsymbol{y}). \tag{9}$$

It can be recognized that the first term in Eq. (9) is a local term which is effective only for points near to the body's boundary. Actually, for points $\boldsymbol{x}$ far from the boundary $\Omega_r(\boldsymbol{x}) \to \Omega_\infty$ and then the first boundary correction term vanishes. The second term in Eq. (9) is the classical nonlocal symmetric term which is expected for unbounded solids. In order to better realize the combined effects of the two terms of Eq. (9), Fig. 1 reports, for a one-dimensional bar of fixed length in uniform state of damage, the two distinct contributions; namely the first local term and the second nonlocal one.

Finally, the kernel defined in Eq. (8) is symmetric everywhere and, incidentally, coincides with the classical one, defined in Eq. (4) for unbounded solids or for the points of bounded bodies sufficiently far (compared to the internal length ℓ_0 or R) from the boundaries. Other relevant properties of the assumption of Eq. (8) will be outlined in the following Sections.

Remark 1. A different view of the nonlocal relation of Eq. (8) can be obtained following a recent observation of Polizzotto (2002). It is in fact possible to rewrite Eq. (8) in the following form

$$\bar{d}(\boldsymbol{x}) = d(x) + \frac{1}{\Omega_\infty}\int_V \alpha(\boldsymbol{x},\boldsymbol{y})[d(\boldsymbol{y}) - d(\boldsymbol{x})]\,\mathrm{d}V(\boldsymbol{y}). \tag{10}$$

This view is somehow also related to recent nonlocal plasticity models of Borino and Failla (2000) (see also Jirásek and Rolshoven (in press) for a quite complete review on nonlocal plasticity models and related consequences). Eq. (10) shows that the local damage field, $d(\boldsymbol{x})$, is corrected by a nonlocal value which is obtained as a spatial average of the excess damage with respect to the damage value at the field point. It transpires that the second nonlocal term is fully inactive for spatial uniform damage fields. It is also quite interesting to observe that Eq. (10) is a real counterpart of the gradient formulations where the local field is enriched by a spatial gradient term which is zero when there is no spatial variations.

3. Thermodynamic framework

In order to ensure that the nonlocal damage formulation does comply with thermodynamic principles, let us assume the existence of a material Helmholtz free energy of the form

$$\psi(\boldsymbol{\varepsilon}, \bar{d}, \xi) = \psi_e(\boldsymbol{\varepsilon}, \bar{d}) + \psi_{\text{in}}(\xi), \tag{11}$$

where $\psi_e(\boldsymbol{\varepsilon}, \bar{d})$ is the damage elastic strain energy that in case of linear isotropic damage models reads

$$\psi_e(\boldsymbol{\varepsilon}, \bar{d}) = \tfrac{1}{2}(1 - \bar{d})\boldsymbol{\varepsilon} : \boldsymbol{D}^{\text{e}} : \boldsymbol{\varepsilon}. \tag{12}$$

In Eq. (11) $\psi_{\text{in}}(\xi)$ is the internal component of the Helmholtz free energy, i.e. the part of energy stored in the micro-structure related to the changes of the material internal properties (i.e. damage strength). ξ is a scalar kinematic internal variable that describes the damage hardening state. In the present formulation the damage variable is treated as nonlocal, but it would be feasible to operate different choices (Jirásek, 1998), as for instance Benvenuti et al. (2000, 2002) chose as nonlocal variable the internal variable ξ.

The integral regularization operation expressed in Eq. (2) can be formally written as

$$\bar{d}(\boldsymbol{x}) = \mathscr{R}(d), \tag{13}$$

where $\mathscr{R}$ is the integral regularization operator applied to the damage field $d(\boldsymbol{x})$. In order to satisfy the second thermodynamics principle, in the present nonlocal context, the Clausius–Duhem inequality must be written in global form, i.e. extended to the entire body V, so that the overall instantaneous body energy dissipation W is

$$W = \int_V (\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi})\,\mathrm{d}V \geqslant 0. \tag{14}$$

Following the approach of Edelen and Laws (1971), Edelen et al. (1971), Polizzotto et al. (1998) and Borino et al. (1999), Eq. (14) can be rewritten in a pointwise form, only after having introduced a nonlocality residual function $P(\boldsymbol{x})$, which takes into account the energy exchanges between neighbour particles. Then the intrinsic dissipation at a given point is

$$D = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi} + P \geqslant 0 \quad \text{in } V. \tag{15}$$

Being the solid a thermodynamically isolated system with reference to the interchange of energies described by P, the following insulation condition holds

$$\int_V P\,\mathrm{d}V = 0. \tag{16}$$

Expanding Eq. (15), taking into account Eqs. (11) and (12), it is

$$D = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial\psi}{\partial\bar{d}}\dot{\bar{d}} - \frac{\partial\psi_{\text{in}}}{\partial\xi}\dot{\xi} + P \geqslant 0. \tag{17}$$

Eq. (17) must hold for any admissible deformation mechanism, either nondissipative elastic or irreversible damaging one. Then, following well established procedures (see for instance Lemaitre and Chaboche (1990)) the following state laws are obtained

$$\boldsymbol{\sigma} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} = (1 - \bar{d})\boldsymbol{D}^{\text{e}} : \boldsymbol{\varepsilon}, \tag{18}$$

$$Y := -\frac{\partial\psi}{\partial\bar{d}} = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{D}^{\text{e}} : \boldsymbol{\varepsilon}, \tag{19}$$

$$\chi := \frac{\partial\psi_{\text{in}}}{\partial\xi}, \tag{20}$$

where Eq. (18) is the stress–strain relation as defined in Eq. (1). Eq. (19) defines the energy release rate Y as the thermodynamic conjugate force of the nonlocal damage $\bar{d}$. Finally, Eq. (20) defines, by means of the internal free energy ψ_{in}, the internal variable χ conjugate of ξ.

By substituting Eqs. (18)–(20) into Eq. (17), the following explicit form for the dissipation function is achieved

$$D = Y\dot{\bar{d}} - \chi\dot{\xi} + P = Y\mathscr{R}(\dot{d}) - \chi\dot{\xi} + P \geqslant 0, \tag{21}$$

where the formal definition of nonlocal regularization operator of Eq. (13) has been used.

3.1. Nonlocality residual function

At every point where an irreversible damage mechanism develops we can always think that the local dissipation must be driven by the local fluxes $\dot{d}$ and $\dot{\xi}$, then it must be possible to write the dissipation in the following bilinear form:

$$D = X\dot{d} - \chi\dot{\xi} \geqslant 0 \quad \text{in } V, \tag{22}$$

where χ has been defined in Eq. (20), whereas X is a (nonlocal) variable thermodynamically conjugated to the local damage variable d and it is now needed to identify its structure. First of all X is nonlocal since it must represent the energy exchanges that neighbour particles interchange during the damage mechanism development. By comparing Eq. (21) with Eq. (22) the expression of the nonlocality residual function is obtained as

$$P = X\dot{d} - Y\mathscr{R}(\dot{d}) \quad \text{in } V. \tag{23}$$

Next, employing the insulation condition of Eq. (16) it is

$$\int_V [X\dot{d} - Y\mathscr{R}(\dot{d})]\,\mathrm{d}V = 0. \tag{24}$$

With reference to the second term of Eq. (24) it can be easily verified that the following Green-type identity holds

$$\int_V Y\mathscr{R}(\dot{d})\,\mathrm{d}V = \int_V \mathscr{R}(Y)\dot{d}\,\mathrm{d}V. \tag{25}$$

It is remarkable that, because of the symmetry of the function $W(\boldsymbol{x},\boldsymbol{y})$ of Eq. (8), the same regularization operator apply also to the conjugate variable Y. This was not the case for the standard nonsymmetric weight function defined in Eq. (4), as shown in Comi and Perego (2001b,c) and Benvenuti et al. (2000, 2002), where the adjoint operator $\mathscr{R}^*$ was applied to Y with the consequent problems of lack of preservation of uniform fields near the boundary as remarked by Comi and Perego (2001b). It can be stated that the symmetry of the weight function implies a self adjoint integral regularization operator with the beneficial properties that will be shown in the following.

Substituting the identity (25) in Eq. (24) gives

$$\int_V [X - \mathscr{R}(Y)]\dot{d}\,\mathrm{d}V = 0, \tag{26}$$

which, being true for any possible damage mechanism field $\dot{d}(\boldsymbol{x})$, allows the identification of the variable X as the integral regularization of the local energy release rate Y, namely

$$X = \bar{Y} \equiv \mathscr{R}(Y). \tag{27}$$

In explicit form, Eq. (27) becomes

$$\bar{Y}(\boldsymbol{x}) = \left(1 - \frac{\Omega_{\mathrm{r}}(\boldsymbol{x})}{\Omega_\infty}\right)Y(\boldsymbol{x}) + \frac{1}{\Omega_\infty}\int_V \alpha(\boldsymbol{x},\boldsymbol{y})Y(\boldsymbol{y})\,\mathrm{d}V(\boldsymbol{y}). \tag{28}$$

Once defined the relation between X and Y, an explicit relation for the nonlocality residual function of Eq. (23) and for the intrinsic dissipation function of Eq. (22) can be obtained, namely

$$P = \mathscr{R}(Y)\dot{d} - Y\mathscr{R}(\dot{d}) \text{ in } V, \tag{29}$$

$$D = \mathscr{R}(Y)\dot{d} - \chi\dot{\xi} \geqslant 0 \text{ in } V. \tag{30}$$

Finally, it can be observed that the above developed nonlocal thermodynamics reasoning closely follows an analogous procedure previously developed in the context of nonlocal plasticity models (Borino et al., 1999; Borino and Failla, 2000).

4. Nonlocal damage activation function and associative damage flow laws

On the basis of the expression of the dissipation function of Eq. (30), it is assumed the existence of a damage activation function $\phi(\bar{Y}, \chi)$ which depends on the variables $\bar{Y} = \mathscr{R}(Y)$ and χ since these are the thermodynamic forces associated to the fluxes $\dot{d}$ and $\dot{\xi}$ in the dissipation function. Under the hypothesis of generalized associative damage behaviour, the following relations can be written:

$$\phi(\bar{Y}, \chi) \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda}\phi = 0 \text{ in } V, \tag{31}$$

$$\dot{d} = \frac{\partial\phi(\bar{Y}, \chi)}{\partial\bar{Y}}\dot{\lambda}, \quad \dot{\xi} = -\frac{\partial\phi(\bar{Y}, \chi)}{\partial\chi}\dot{\lambda} \text{ in } V. \tag{32}$$

These are the usual laws for damage generalized standard materials, i.e. ϕ besides being the damage activation function, plays also the role of damage potential function for $\dot{d}$ and $\dot{\xi}$. Eqs. (31) and (32) may appear fully equivalent to the standard local damage relations, however it must be observed that the presence of the variable $\bar{Y} = \mathscr{R}(Y)$ makes Eqs. (31) and (32) a system of relations spatially coupled and this is the remarkable feature of nonlocal damage models.

In the following, without loosing generality, the damage activation function is written in the rather simple form that follows:

$$\phi(\bar{Y}, \chi) = \bar{Y} - \chi - Y_0 \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda}\phi = 0 \text{ in } V \tag{33}$$

and consequently the flow laws reads

$$\dot{d} = \frac{\partial\phi(\bar{Y}, \chi)}{\partial\bar{Y}}\dot{\lambda} = \dot{\lambda}, \quad \dot{\xi} = -\frac{\partial\phi(\bar{Y}, \chi)}{\partial\chi}\dot{\lambda} = \dot{\lambda} \text{ in } V. \tag{34}$$

From Eq. $(33)_1$ the role played by χ as a damage hardening variable appears clear, whereas Y_0 is the initial threshold for the first damage increment occurrence.

4.1. Principle of maximum damage dissipation

Once the associative behaviour has been postulated, it is then possible to define a *principle of maximum damage dissipation*. This principle can be viewed as a direct extension to the damage context of a well known principle of the plasticity theory (Simo and Hughes, 1998). The principle asserts that the material state corresponding to an assigned damage mechanism is the one that maximize the intrinsic dissipation energy (or net entropy production). If the mechanism is defined by the damage flow fields $\dot{d}$ and $\dot{\xi}$ diffuse in V, then the principle reads

$$\max_{(\bar{Y},\chi)} \int_V (\bar{Y}\dot{d} - \chi\dot{\xi})\,\mathrm{d}V \quad \text{subject to}: \ \phi(\bar{Y},\chi) \leqslant 0 \ \text{in} \ V. \tag{35}$$

Problem (35) is analogous to the Hill theorem of the local associative plasticity, with the difference that it is written in a global form, i.e. with reference to the entire domain V. Following standard procedures of the optimization theory, it is possible to show that the Khun–Tucker conditions of the maximization problem (35) are Eqs. (33) and (34). The nonlocal feature of problem (35) is hidden in the subsequent step aimed to obtain the local value Y from the integral relation $\bar{Y} = \mathscr{R}(Y)$, where $\bar{Y}$ is given by the solution of problem (35).

An alternative, and perhaps more interesting, form of the maximum damage dissipation principle can be derived assuming as dissipative fluxes $\dot{\bar{d}}$ and $\dot{\xi}$. In this case the principle is written as

$$\max_{(Y,\chi)} \int_V (Y\dot{\bar{d}} - \chi\dot{\xi})\,\mathrm{d}V \quad \text{subject to}: \ \phi = \mathscr{R}(Y) - \chi - Y_0 \leqslant 0 \ \text{in} \ V. \tag{36}$$

Looking for the extremal conditions by means of the Lagrangian multiplier method, the Lagrangian functional can be written

$$L = -\int_V (Y\dot{\bar{d}} - \chi\dot{\xi})\,\mathrm{d}V + \int_V \dot{\lambda}(\mathscr{R}(Y) - \chi - Y_0)\,\mathrm{d}V, \tag{37}$$

where $\dot{\lambda} \geqslant 0$ is the relevant Lagrangian multiplier. Then, the first variation of the Lagrangian of Eq. (37) reads

$$\delta L = -\int_V (\delta Y\dot{\bar{d}} - \delta\chi\dot{\xi})\,\mathrm{d}V + \int_V \dot{\lambda}(\mathscr{R}(\delta Y) - \delta\chi)\,\mathrm{d}V + \int_V \delta\dot{\lambda}\phi\,\mathrm{d}V. \tag{38}$$

Considering that, by Eq. (25), the following identity holds

$$\int_V \dot{\lambda}\mathscr{R}(\delta Y)\,\mathrm{d}V = \int_V \mathscr{R}(\dot{\lambda})\delta Y\,\mathrm{d}V, \tag{39}$$

Eq. (38) reduces to

$$\delta L = \int_V \delta Y(\mathscr{R}(\dot{\lambda}) - \dot{\bar{d}})\,\mathrm{d}V + \int_V \delta\chi(\dot{\xi} - \dot{\lambda})\,\mathrm{d}V + \int_V \delta\dot{\lambda}\phi\,\mathrm{d}V, \tag{40}$$

which produce as extremal conditions

$$\dot{\bar{d}} = \mathscr{R}(\dot{\lambda}), \quad \dot{\lambda} = \dot{\xi} \ \text{in} \ V, \tag{41}$$

$$\phi(\mathscr{R}(Y),\chi) \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \phi\dot{\lambda} = 0 \ \text{in} \ V. \tag{42}$$

From Eq. $(42)_1$ follows that $\dot{\lambda} = \dot{d}$.

5. Material rate response

In this section the material response to an assigned strain rate field, $\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x})$, is investigated. The material, considered as an ensemble of continuum material points, is in a known initial state characterized by $\boldsymbol{\sigma}(\boldsymbol{x})$, $\boldsymbol{\varepsilon}(\boldsymbol{x})$, $Y(\boldsymbol{x})$, $d(\boldsymbol{x})$, $\chi(\boldsymbol{x})$; also let $V_d \subseteq V$ be the region where $\phi(\bar{Y},\chi) = 0$. The material response is elastic (and of local type) in $V_e = V - V_d$, elasto-damaging in V_d where it must be $\dot{\phi} \leqslant 0$, $\dot{\lambda} \geqslant 0$ and $\dot{\phi}\dot{\lambda} = 0$. Expanding the damage activation function in its rate form we have

$$\dot{\phi}(\mathscr{R}(Y),\chi) = \frac{\partial\phi}{\partial\mathscr{R}(Y)}\mathscr{R}(\dot{Y}) + \frac{\partial\phi}{\partial\chi}\dot{\chi} = \mathscr{R}(\dot{Y}) - \dot{\chi} \leqslant 0 \text{ in } V_d. \tag{43}$$

The state laws of Eqs. (18)–(20), rewritten in rate form, reads

$$\dot{Y} = \boldsymbol{\varepsilon} : \boldsymbol{D}^{\mathrm{e}} : \dot{\boldsymbol{\varepsilon}} \equiv \dot{\ell}_\varepsilon; \quad \dot{\chi} = h(\xi)\dot{\xi} = h(\xi)\dot{\lambda}, \tag{44}$$

where the definition $h(\xi) \equiv \partial^2\psi_{\mathrm{in}}/\partial\xi^2$ has been adopted. Substituting Eq. (44) into Eq. (43) the complete incremental damage problem is obtained as

$$\dot{\phi} = R(\dot{\ell}_\varepsilon) - h(\xi)\dot{\lambda} \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda}\dot{\phi} = 0 \text{ in } V_d. \tag{45}$$

The complementarity problem expressed by Eq. (45) admits a variational representation by a minimum principle as the following

$$\min_{\dot{\lambda}} \Pi[\dot{\lambda}] := \int_{V_d} \left[\frac{1}{2}h(\xi)\dot{\lambda}^2 - \mathscr{R}(\dot{\ell}_\varepsilon)\dot{\lambda}\right] \mathrm{d}V, \quad \text{subject to } \dot{\lambda} \geqslant 0 \text{ in } V_d. \tag{46}$$

The stationarity conditions of problem (46) give the damage multiplier field $\dot{\lambda}(\boldsymbol{x}) \geqslant 0$ that satisfies Eq. (45). The term $h(\xi) \equiv \partial^2\psi_{\mathrm{in}}/\partial\xi^2$ is positive for every value of ξ, since it represents a damage hardening modulus, i.e. it is expected an increase of the damage elastic domain (in the strain space) as damage develops. The positiveness of the coefficient of the quadratic functional (46) ensures that the second variation of the functional, i.e. $\delta^2\Pi = \int_{V_d} h(\xi)(\delta\dot{\lambda})^2\,\mathrm{d}V$ is positive and then the functional is convex. Finally, the convexity of the functional guaranties the existence and the uniqueness of the solution field $\dot{\lambda}(\boldsymbol{x})$.

Remark 2. At difference of the most standard nonlocal formulations (Strömberg and Ristimnaa, 1996; Borino et al., 1999; Borino and Failla, 2000; Benvenuti et al., 2000, 2002), the truly nonlocality features in Eqs. (45) and (46) are carried by $\mathscr{R}(\dot{\ell}_\varepsilon)$, that is

$$\mathscr{R}(\dot{\ell}_\varepsilon) = \left(1 - \frac{\Omega_{\mathrm{r}}(\boldsymbol{x})}{\Omega_\infty}\right)\boldsymbol{\varepsilon}(\boldsymbol{x}) : \boldsymbol{D}^{\mathrm{e}} : \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) + \frac{1}{\Omega_\infty}\int_V \alpha(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\varepsilon}(\boldsymbol{y}) : \boldsymbol{D}^{\mathrm{e}} : \dot{\boldsymbol{\varepsilon}}(\boldsymbol{y})\,\mathrm{d}V(\boldsymbol{y}), \tag{47}$$

which depends on the assigned driving strain rate field, $\dot{\boldsymbol{\varepsilon}}(\boldsymbol{x})$. Eq. (47), once computed at every point of the body, produces a problem which is formally equivalent to the classical local damage loading problem. In particular the solution of Eq. (45) or Eq. (46) is

$$\dot{\lambda} = \frac{1}{h(\xi)}\langle\mathscr{R}(\dot{\ell}_\varepsilon)\rangle \text{ in } V, \tag{48}$$

where the MacAuley operator, $\langle x\rangle = (x + |x|)/2$ has been used.

Remark 3. The very special feature outlined in the previous Remark is rooted on the fact that when the damage is chosen as nonlocal variable, the relations derived for linear isotropic damage laws are much simpler than the one obtained choosing as nonlocal variable the internal variable ξ (as in Benvenuti et al., 2000, 2002) and even simpler than the most used nonlocal plasticity models (Strömberg and Ristimnaa, 1996; Borino et al., 1999; Borino and Failla, 2000; Jirásek and Rolshoven, in press), where a fully nonlocal coupling is present. The remarkable simplification, already present in the original model of Pijaudier-Cabot and Bažant (1987), is only due to the linearity of the isotropic damage model, which, through the state equation (19), makes Y not dependent on damage, but only on the local strain $\boldsymbol{\varepsilon}(\boldsymbol{x})$. If, for instance, a quadratic damage law were adopted, then the damage state equation (19) would express Y as a function of the nonlocal damage $\bar{d} = \mathscr{R}(d)$, hence an integral term in $\dot{\lambda}(\boldsymbol{x})$ would be present in Eq. (45). The problem

(46) would be an integral variational theorem and finally Eq. (48) would become a Fredholm integral equation in the unknown field $\dot{\lambda}(\boldsymbol{x})$, to be solved following more complex strategies, similar to those adopted in Borino et al. (1999).

5.1. Nonlocal continuum tangent moduli tensor

The incremental constitutive law can be effectively expressed by means of stress–strain rate relations in which the continuum tangent operator is defined. Let us write the stress–strain relation of Eq. (1) in rate form

$$\dot{\boldsymbol{\sigma}}(\boldsymbol{x}) = (1 - \bar{d}(\boldsymbol{x}))\boldsymbol{D}^{\mathrm{e}} : \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) - \boldsymbol{D}^{\mathrm{e}} : \boldsymbol{\varepsilon}(\boldsymbol{x})\dot{\bar{d}}(\boldsymbol{x}). \tag{49}$$

Substituting the definition of Eq. (9) in Eq. (49) for $\bar{d}$, the following relation is obtained

$$\dot{\boldsymbol{\sigma}}(\boldsymbol{x}) = (1 - \bar{d}(\boldsymbol{x}))\boldsymbol{D}^{\mathrm{e}} : \dot{\boldsymbol{\varepsilon}}(\boldsymbol{x}) - \boldsymbol{D}^{\mathrm{e}} : \boldsymbol{\varepsilon}(\boldsymbol{x})\left\{\left(1 - \frac{\Omega_{\mathrm{r}}(\boldsymbol{x})}{\Omega_\infty}\right)\dot{\lambda}(\boldsymbol{x}) + \frac{1}{\Omega_\infty}\int_V \alpha(\boldsymbol{x},\boldsymbol{y})\dot{\lambda}(\boldsymbol{y})\,\mathrm{d}V(\boldsymbol{y})\right\}. \tag{50}$$

Next, substituting $\dot{\lambda}$ from Eq. (48) in Eq. (50), the following integral continuum relation follows:

$$\dot{\boldsymbol{\sigma}}(\boldsymbol{x}) = \int_V \boldsymbol{D}^{\mathrm{ed}} : \dot{\boldsymbol{\varepsilon}}(\boldsymbol{y})\,\mathrm{d}V(\boldsymbol{y}), \tag{51}$$

where $\boldsymbol{D}^{\mathrm{ed}}(\boldsymbol{x},\boldsymbol{y})$ is the nonlocal continuum tangent operator having the following structure

$$\boldsymbol{D}^{\mathrm{ed}}(\boldsymbol{x},\boldsymbol{y}) = (1 - \bar{d}(\boldsymbol{x}))\boldsymbol{D}^{\mathrm{e}}\delta(\boldsymbol{x},\boldsymbol{y}) - \{A_{\mathrm{L}}(\boldsymbol{x},\boldsymbol{y}) + A_{\mathrm{NL}}(\boldsymbol{x},\boldsymbol{y})\}[(\boldsymbol{D}^{\mathrm{e}} : \boldsymbol{\varepsilon}(\boldsymbol{x})) \otimes (\boldsymbol{\varepsilon}(\boldsymbol{y}) : \boldsymbol{D}^{\mathrm{e}})], \tag{52}$$

where the following definitions have been adopted

$$A_{\mathrm{L}}(\boldsymbol{x},\boldsymbol{y}) = \left(1 - \frac{\Omega_{\mathrm{r}}(\boldsymbol{x})}{\Omega_\infty}\right)L(\boldsymbol{x})\delta(\boldsymbol{x},\boldsymbol{y}), \tag{53}$$

$$A_{\mathrm{NL}}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\Omega_\infty}(L(\boldsymbol{x}) + L(\boldsymbol{y}))\alpha(\boldsymbol{x},\boldsymbol{y}) + \frac{1}{\Omega_\infty^2}\beta(\boldsymbol{x},\boldsymbol{y}) \tag{54}$$

with

$$L(\boldsymbol{x}) = \frac{1}{h(\boldsymbol{x})}\left(1 - \frac{\Omega_{\mathrm{r}}(\boldsymbol{x})}{\Omega_\infty}\right), \tag{55}$$

$$\beta(\boldsymbol{x},\boldsymbol{y}) = \int_V \frac{1}{h(\boldsymbol{z})}\alpha(\boldsymbol{x},\boldsymbol{z})\alpha(\boldsymbol{z},\boldsymbol{y})\,\mathrm{d}V(\boldsymbol{z}). \tag{56}$$

It can be easily verified that the presented formulation conducts to a nonlocal continuum tangent moduli tensor that enjoys the relevant property $\boldsymbol{D}^{\mathrm{ed}}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{D}^{\mathrm{ed}}(\boldsymbol{y},\boldsymbol{x})$.

It is also interesting to observe that the nonlocal continuum tangent moduli tensor of Eq. (52) is obtained as the sum of two terms. The first term is the secant moduli tensor, whereas the second term is obtained as sum of two more contributions. The first contribution, related to $A_{\mathrm{L}}(\boldsymbol{x},\boldsymbol{y})$, gives the local damage loading effect; the second contribution, related to $A_{\mathrm{NL}}(\boldsymbol{x},\boldsymbol{y})$, is concerned with the nonlocal damage loading conditions, i.e. the effects which are promoted by the damage loading of neighbour points.

6. Structural rate response

The elastic-damage solid body occupying the domain $V \in \mathbb{R}^3$, with boundary $\partial V = S = S_U \cup S_T$, constitutively analyzed in the previous Sections, is subjected to body forces $\boldsymbol{b}$ in V, surface tractions $\boldsymbol{t}$ on the free surface S_T and to imposed displacements $\boldsymbol{u}^*$ on the constrained surface S_U. All the external actions vary in a quasi-static manner and cause the body to undergo small displacements and strains. We are now interested to the response of the body to a given load rates as $\dot{\boldsymbol{b}}$ in V, $\dot{\boldsymbol{t}}$ on S_T and $\dot{\boldsymbol{u}}^*$ on S_U. Assuming known the initial state, it is possible to identify the region $V_d \subseteq V$ where the damage limit condition has been attained (i.e. where $\phi = 0$), whereas $\phi < 0$ in the complementary region, V_d^c. The equations governing the structural rate-type nonlocal boundary value problem read

$$\dot{\boldsymbol{\varepsilon}} = \nabla^s \dot{\boldsymbol{u}} \text{ in } V, \quad \dot{\boldsymbol{u}} = \dot{\boldsymbol{u}}^* \text{ on } S_U, \tag{57}$$

$$\nabla \cdot \dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{b}} = \boldsymbol{0} \text{ in } V, \quad \dot{\boldsymbol{\sigma}} \cdot \boldsymbol{n} = \dot{\boldsymbol{t}} \text{ on } S_T, \tag{58}$$

$$\dot{\boldsymbol{\sigma}} = (1 - \bar{d})\boldsymbol{D}^{\mathrm{e}} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{D}^{\mathrm{e}} : \boldsymbol{\varepsilon}\dot{\bar{d}} \text{ in } V, \tag{59}$$

$$\dot{\chi} = h(\xi)\dot{\xi}, \quad \dot{Y} = \boldsymbol{\varepsilon} : \boldsymbol{D}^{\mathrm{e}} : \dot{\boldsymbol{\varepsilon}}, \quad [\dot{\bar{Y}} = \mathscr{R}(\dot{Y})] \text{ in } V, \tag{60}$$

$$\dot{\phi} = \dot{\bar{Y}} - \dot{\chi} \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda}\dot{\phi} = 0 \text{ in } V_d, \tag{61}$$

$$\dot{d} = \dot{\xi} = \dot{\lambda}, \quad \dot{\bar{d}} = \mathscr{R}(\dot{\lambda}) \text{ in } V. \tag{62}$$

In the previous relations, $\boldsymbol{n}$ denotes the unit external normal to S and ∇^s is the symmetric part of the gradient operator ∇. Eq. (57) are the compatibility relations, Eq. (58) are the equilibrium relations, Eqs. (59) and (60) are the rate-form state equations and finally Eqs. (61) and (62) are the damage loading/unloading (or evolutive) laws.

The structural rate response can be obtained solving the set of relations (57)–(62). As it always happen in solid mechanics, the actual closed form solution can be achieved only for very few simple problems and in order to achieve practical solutions a finite increment approach together with a finite element space discretization is adopted. In this view it is of interest to show that problem (57)–(62) can be characterized by the following kinematic-type variational principle.

$$\begin{aligned}\mathscr{L}[\dot{\boldsymbol{u}}, \dot{\lambda}] = &\int_V \left\{ \frac{1}{2} \nabla^s \dot{\boldsymbol{u}} : (1 - \bar{d})\boldsymbol{D}^{\mathrm{e}} : \nabla^s \dot{\boldsymbol{u}} - \mathscr{R}(\dot{\lambda}) \nabla^s \boldsymbol{u} : \boldsymbol{D}^{\mathrm{e}} : \nabla^s \dot{\boldsymbol{u}} \right\} \mathrm{d}V + \frac{1}{2} \int_V h(\xi) \dot{\lambda}^2 \,\mathrm{d}V \\ &- \int_V \dot{\boldsymbol{b}} \cdot \dot{\boldsymbol{u}} \,\mathrm{d}V - \int_{S_T} \dot{\boldsymbol{t}} \cdot \dot{\boldsymbol{u}} \,\mathrm{d}S,\end{aligned} \tag{63}$$

where the unknown kinematic fields $\dot{\boldsymbol{u}}, \dot{\lambda}$ satisfy the following constraint equations

$$\dot{\boldsymbol{u}} = \dot{\boldsymbol{u}}^* \quad \text{on } S_U, \tag{64}$$

$$\dot{\lambda} \geqslant 0 \text{ in } V_d, \quad \dot{\lambda} = 0 \text{ in } V/V_d, \tag{65}$$

The variational principle states that the constrained fields $\dot{\boldsymbol{u}}$, $\dot{\lambda}$ that make $\mathscr{L}$ stationary together with some related $\dot{\boldsymbol{\varepsilon}}$, $\dot{\boldsymbol{\sigma}}$, $\dot{Y}$, $\dot{\chi}$, solve the structural rate problem of Eqs. (57)–(62) and conversely the solution ($\dot{\boldsymbol{u}}$, $\dot{\lambda}$, $\dot{\boldsymbol{\varepsilon}}$, $\dot{\boldsymbol{\sigma}}$, $\dot{Y}$, $\dot{\chi}$) to the latter rate problem is such that the subset $\dot{\boldsymbol{u}}, \dot{\lambda}$ makes $\mathscr{L}$ stationary.

In order to prove the variational principle, let us first consider the augmented Lagrangian functional

$$\mathscr{L}_a[\dot{\boldsymbol{u}}, \dot{\lambda}, \dot{\boldsymbol{p}}] = \mathscr{L}[\dot{\boldsymbol{u}}, \dot{\lambda}] - \int_{S_U} \dot{\boldsymbol{p}} \cdot (\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^*) \,\mathrm{d}S, \tag{66}$$

where $\dot{\boldsymbol{p}}$ is the appropriate (rate traction-like) Lagrangian multiplier. The first variation of $\mathscr{L}_a$, after the application of the divergence theorem and of the Green-type identity observed in Eq. (39), proves to be

$$\begin{aligned}\delta\mathscr{L}_a = &-\int_V \{\nabla\cdot[(1-\bar{d})\boldsymbol{D}^{\mathrm{e}}:\nabla^s\dot{\boldsymbol{u}} - \mathscr{R}(\dot{\lambda})\boldsymbol{D}^{\mathrm{e}}:\nabla^s\boldsymbol{u}] + \dot{\boldsymbol{b}}\}\cdot\delta\dot{\boldsymbol{u}}\,\mathrm{d}V \\ &- \int_V \{\mathscr{R}(\nabla^s\boldsymbol{u}:\boldsymbol{D}^{\mathrm{e}}:\nabla^s\dot{\boldsymbol{u}}) - h(\xi)\dot{\lambda}\}\delta\dot{\lambda}\,\mathrm{d}V + \int_{S_T}\{[(1-\bar{d})\boldsymbol{D}^{\mathrm{e}}:\nabla^s\dot{\boldsymbol{u}} - \mathscr{R}(\dot{\lambda})\boldsymbol{D}^{\mathrm{e}}:\nabla^s\boldsymbol{u}]\cdot\boldsymbol{n} - \dot{\boldsymbol{t}}\}\cdot\delta\dot{\boldsymbol{u}}\,\mathrm{d}S \\ &+ \int_{S_U}\{[(1-\bar{d})\boldsymbol{D}^{\mathrm{e}}:\nabla^s\dot{\boldsymbol{u}} - \mathscr{R}(\dot{\lambda})\boldsymbol{D}^{\mathrm{e}}:\nabla^s\boldsymbol{u}]\cdot\boldsymbol{n} - \dot{\boldsymbol{p}}\}\cdot\delta\dot{\boldsymbol{u}}\,\mathrm{d}S + \int_{S_U}\{\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}^*\}\cdot\delta\dot{\boldsymbol{p}}\,\mathrm{d}S.\end{aligned} \tag{67}$$

The conditions for which $\delta\mathscr{L}_a = 0$ for any arbitrary admissible variations, that are the Euler-Lagrange equations of the variational principle (63), turn out to be equivalent to the rate structural problem, Eqs. (57)–(62), with the Lagrangian multiplier vector $\dot{\boldsymbol{p}}$ being identified as the reactive traction on S_u. In other words, the stationary solution solves the structural rate problem of Eqs. (57)–(62).

The converse part of the theorem can be proved by observing that the fields $(\dot{\boldsymbol{u}}, \dot{\lambda})$ pertaining to the solution to the structural rate problem, together with $\dot{\boldsymbol{p}} = \dot{\boldsymbol{\sigma}}\cdot\boldsymbol{n}$ on S_U, make $\delta\mathscr{L}_a$ of Eq. (67) vanishes for arbitrary variations and thus $\mathscr{L}(\dot{\boldsymbol{u}}, \dot{\lambda})$ is stationary.

7. Finite element incremental problem

As every continuum nonholonomic dissipative problem, the practical resolution of the rate problem requires two different discretization techniques. The first is related to the integration along the loading path of the rate constitutive equations, and this is usually achieved by the Euler backward difference scheme. The second discretization is the space discretization and typically is obtained using finite element techniques. Since, in this specific nonlocal context, another spatial integration procedure has to be carried out, it is very convenient to use the same finite element mesh and the related Gauss points for performing also the nonlocal variables evaluation.

The standard relation for the finite element discretized structural equilibrium at the end of the loading step, $(n, n+1)$, reads

$$\boldsymbol{F}_{\mathrm{int}}^{n+1} = \int_V \boldsymbol{B}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\sigma}^{n+1}(\boldsymbol{x})\,\mathrm{d}V, \tag{68}$$

where $\boldsymbol{F}_{\mathrm{int}}^{n+1}$ is the nodal load vector computed at the end of the step, $\boldsymbol{B}(\boldsymbol{x})$ is the strain–displacement operator and $\boldsymbol{\sigma}^{n+1}$ is the stress vector computed at the end of the step. The integral is then approximated by a sum over the Gauss integration points of the elements as

$$\boldsymbol{F}_{\mathrm{int}}^{n+1} = \sum_{e=1}^{N_e}\sum_{ge=1}^{N_{ge}} w_{ge}\boldsymbol{B}^{eT}(\boldsymbol{x}_{ge})\boldsymbol{\sigma}^{n+1}(\boldsymbol{x}_{ge}) = \sum_{g=1}^{N_g} w_g\boldsymbol{B}_g^{\mathrm{T}}\boldsymbol{\sigma}_g^{n+1}, \tag{69}$$

where N_e is the total number of finite elements, N_{ge} and N_g are the number of Gauss points for element and the total number of Gauss points of the structure. $\boldsymbol{x}_{ge}$ is the coordinate vector of the integration points and finally w_g is the Gauss integration weight coefficient. In the last term of Eq. (69) the index g stands for the relevant term computed at the point $\boldsymbol{x}_g$.

The nonlinear system of equations of Eq. (69) is then addressed by a Newton–Raphson iterative scheme consisting in an elastic predictor phase and a nonlocal corrector phase. (The procedure closely follows the classically procedure of computational plasticity, see e.g. Simo and Hughes (1998).) The predictor phase is a linear elastic analysis that produces a displacement vector and then a strain distribution at the end of the step for every Gauss point, $\boldsymbol{\varepsilon}_g^{n+1}$, which is then the driving term for imposing the constitutive consistency in the nonlocal damage corrector phase.

In order to impose the nonlocal damage law at the end of the step the constitutive relations are integrated by the Euler backward scheme, namely

$$\boldsymbol{\sigma}_g^{n+1} = (1 - \bar{d}_g^{n+1})\boldsymbol{D}^{\mathrm{e}}\boldsymbol{\varepsilon}_g^{n+1}, \tag{70}$$

$$\chi_g^{n+1} = \left.\frac{\partial \psi_{\mathrm{in}}}{\partial \xi}\right|_{n+1} \equiv \chi_g^n + f(\Delta\lambda_g), \tag{71}$$

$$Y_g^{n+1} = \tfrac{1}{2}\boldsymbol{\varepsilon}_g^{n+1T}\boldsymbol{D}^{\mathrm{e}}\boldsymbol{\varepsilon}_g^{n+1}, \tag{72}$$

$$\Delta\lambda_g = \Delta d_g = \Delta\xi_g, \tag{73}$$

$$\phi_g^{n+1} = \bar{Y}_g^{n+1} - \chi_g^{n+1} - Y^0 \leqslant 0, \quad \Delta\lambda_g \geqslant 0, \quad \phi_g^{n+1}\Delta\lambda_g = 0, \tag{74}$$

where $f(\Delta\lambda_g)$ is a function containing all the terms in the unknown $\Delta\lambda$. Moreover, the nonlocal integral relations of Eqs. (27) and (2) are computed following the same approximation based on the finite element mesh and Gauss integration rule, namely

$$\bar{Y}_p^{n+1} = \sum_{q=1}^{N_g} w_q W_{pq} Y_q^{n+1} = \frac{1}{2}\sum_{q=1}^{N_g} w_q W_{pq}\boldsymbol{\varepsilon}_q^{n+1T}\boldsymbol{D}^{\mathrm{e}}\boldsymbol{\varepsilon}_q^{n+1}, \tag{75}$$

$$\bar{d}_g^{n+1} = \bar{d}_g^n + \sum_{\ell=1}^{N_g} w_\ell W_{g\ell}\Delta\lambda_\ell. \tag{76}$$

Substituting Eqs. (75), (71) and (72) in Eq. (74), the step consistency condition is obtained, namely

$$\phi_g^{n+1} = \phi_g^{n+1(tr)} - f(\Delta\lambda_g) \leqslant 0, \quad \Delta\lambda_g \geqslant 0, \quad \phi_g^{n+1}\Delta\lambda_g = 0, \tag{77}$$

where the following position holds

$$\phi_g^{n+1(tr)} \equiv \bar{Y}_g^{n+1} - \chi_g^n - Y^0, \tag{78}$$

which has to be considered a trial elastic value. The solution of problem (77) can be easily achieved following a classical approach of computational plasticity (Simo and Hughes, 1998), namely if $\phi_g^{n+1(tr)} > 0$ then the damage multiplier increment is evaluated solving for the unknown $\Delta\lambda_g$ the nonlinear scalar equation $\phi_g^{n+1(tr)} - f(\Delta\lambda_g) = 0$.

Once the local complementarity problem has been solved at every Gauss integration point, the nonlocal damage distribution $\bar{d}_g^{n+1}$, is evaluated by Eq. (76) and then the updated stress (complying damage constitutive equations), $\boldsymbol{\sigma}_g^{n+1}$, is computed by Eq. (70). Finally this stress is inserted into Eq. (69) and the resulting out of balance force is utilized for a new elastic predictor phase of the iterative procedure.

It is remarkable that the damage correction phase is performed at each integration point and that the nonlocal nature of the problem is displayed only by the fact that the damage driving force $\bar{Y}_g^{n+1}$ is trivially computed by means of Eq. (75) using the (known) strains of all the neighbour Gauss points. This simplification is due only to the choice of the damage variable as nonlocal variable and also to the fact that a simple linear isotropic damage law has been adopted. Usually the damage corrector phase requires a more complicated nonlocal iteration procedure (Benvenuti et al., 2000, 2002).

7.1. Symmetric consistent tangent stiffness matrix

The predictor phase of the Newton–Raphson iterative scheme can be performed by using the initial stiffness matrix, the secant matrix, or alternatively by means of the tangent stiffness matrix obtained by linearizing consistently with the Euler backward difference scheme, the nonlinear equilibrium relations. It is well known that the consistent tangent stiffness matrix significantly reduces the number of iterations ensuring a quadratic asymptotic rate of convergence (Simo and Hughes, 1998). In the following, the procedure originally proposed by Jirásek (2001), Patzák and Jirásek (2001) and Jirásek and Rolshoven (in press) and applied also by Comi and Perego (2001b,c), is utilized in order to derive the consistent tangent stiffness matrix.

Let us rewrite the finite element discretized equilibrium equation (69) in the following equivalent form

$$\boldsymbol{F}_{\text{int}} = \sum_{p=1}^{N_g} w_p \boldsymbol{B}_p^{\text{T}} (1 - \bar{d}_p) \boldsymbol{D}^{\text{e}} \boldsymbol{\varepsilon}_p, \tag{79}$$

where the superscript $n+1$ has been omitted, all the quantities being evaluated at the end of the same loading step. Observing that the strain is related to the structural nodal displacement vector $\boldsymbol{U}$ by the compatibility relation $\boldsymbol{\varepsilon}_p = \boldsymbol{B}_p \boldsymbol{U}$, Eq. (79) transforms into

$$\boldsymbol{F}_{\text{int}} = \sum_{p=1}^{N_g} w_p (1 - \bar{d}_p) \boldsymbol{K}_p^{\text{e}} \boldsymbol{U}, \tag{80}$$

where $\boldsymbol{K}_p^{\text{e}} := \boldsymbol{B}_p^{\text{T}} \boldsymbol{D}^{\text{e}} \boldsymbol{B}_p$ describes the undamaged elastic stiffness contribution of the integration point $\boldsymbol{x}_p$ to the structure elastic stiffness. The tangent stiffness matrix is then defined by the following relation

$$\boldsymbol{K}_{\text{ed}} := \frac{\partial \boldsymbol{F}_{\text{int}}}{\partial \boldsymbol{U}} = \sum_{p=1}^{N_g} w_p (1 - \bar{d}_p) \boldsymbol{K}_p^{\text{e}} - \sum_{p=1}^{N_g} w_p \boldsymbol{K}_p^{\text{e}} \boldsymbol{U} \left(\frac{\partial \bar{d}_p}{\partial \boldsymbol{U}} \right)^{\text{T}}. \tag{81}$$

From the definition of nonlocal damage of Eq. (2), spatially discretized in the form given by Eq. (76), we obtain

$$\frac{\partial \bar{d}_p}{\partial \boldsymbol{U}} = \sum_{q=1}^{N_g^{\text{act}}} w_q W_{pq} \frac{\partial d_q}{\partial \boldsymbol{U}}, \tag{82}$$

where the summation is extended to the Gauss points that are damage active, i.e. where the trial value given by Eq. (78) is greater than zero.

The variation of the damage loading condition for the active points computed at the end of the step reads

$$\delta \phi_q = \left. \frac{\partial \phi}{\partial \bar{Y}} \right|_q \delta \bar{Y}_q + \left. \frac{\partial \phi}{\partial \chi} \right|_q \delta \chi_q = 0. \tag{83}$$

The variation of the internal variable χ can be related, by the state Eq. (20) and the flow law of Eq. $(34)_2$, to the damage multiplier variation $\delta\lambda$ by means

$$\delta \chi_q = - \left. \frac{\partial^2 \psi_{\text{in}}}{\partial \xi^2} \right|_q \left. \frac{\partial \phi}{\partial \chi} \right|_q \delta \lambda_q, \tag{84}$$

which substituted into Eq. (83) allows the damage multiplier variation to be derived as

$$\delta\lambda_q = A_q \frac{\partial\phi}{\partial\bar{Y}}\bigg|_q \delta\bar{Y}_q \quad \left[\text{with } A_q = \left(\frac{\partial\phi}{\partial\bar{Y}}\bigg|_q \frac{\partial^2\psi_{\text{in}}}{\partial\xi^2}\bigg|_q \frac{\partial\phi}{\partial\bar{Y}}\bigg|_q\right)^{-1}\right]. \tag{85}$$

From the damage flow law of Eq. $(34)_1$ follows

$$\delta d_q = \frac{\partial\phi}{\partial\bar{Y}}\bigg|_q \delta\lambda_q = H_q\delta\bar{Y} \quad \left[\text{with } H_q = \left(\frac{\partial\phi}{\partial\bar{Y}}\bigg|_q A_q \frac{\partial\phi}{\partial\bar{Y}}\bigg|_q\right)\right], \tag{86}$$

which, considering the definition of Eq. (27) and its space discretized form of Eq. (75), becomes

$$\delta d_q = H_q \sum_{\ell=1}^{N_g} w_\ell W_{q\ell}\boldsymbol{\varepsilon}_\ell^{\mathrm{T}}\boldsymbol{D}^{\mathrm{e}}\delta\boldsymbol{\varepsilon}_\ell = H_q \sum_{\ell=1}^{N_g} w_\ell W_{q\ell}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{B}_\ell^{\mathrm{T}}\boldsymbol{D}^{\mathrm{e}}\boldsymbol{B}_\ell\delta\boldsymbol{U} = H_q \sum_{\ell=1}^{N_g} w_\ell W_{q\ell}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{K}_\ell^{\mathrm{e}}\delta\boldsymbol{U}. \tag{87}$$

Substituting Eq. (87) in Eq. (82), the following relation is obtained

$$\left(\frac{\partial d_q}{\partial\boldsymbol{U}}\right)^{\mathrm{T}} = H_q \sum_{\ell=1}^{N_g} w_\ell W_{q\ell}\boldsymbol{U}\boldsymbol{K}_\ell^{\mathrm{e}}, \tag{88}$$

such that, substituting Eq. (88) into Eq. (81), the expression of the tangent stiffness matrix becomes

$$\boldsymbol{K}_{\mathrm{ed}} = \sum_{p=1}^{N_g} w_p(1-\bar{d}_p)\boldsymbol{K}_p^{\mathrm{e}} - \sum_{p=1}^{N_g}\sum_{q=1}^{N_g^{\mathrm{act}}}\sum_{\ell=1}^{N_g} w_p w_q w_\ell W_{pq}W_{q\ell}\boldsymbol{K}_p^{\mathrm{e}}\boldsymbol{U}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{K}_\ell^{\mathrm{e}}. \tag{89}$$

Finally, observing that the first term is the secant stiffness matrix $\boldsymbol{K}_{\mathrm{sec}}$, and defining a vector $\boldsymbol{b}_p^\sigma = \boldsymbol{B}_p^{\mathrm{T}}\boldsymbol{D}^{\mathrm{e}}\boldsymbol{B}_p\boldsymbol{U} = \boldsymbol{K}_p^{\mathrm{e}}\boldsymbol{U}$, Eq. (89) can be rewritten in the following equivalent form

$$\boldsymbol{K}_{\mathrm{ed}} = \boldsymbol{K}_{\mathrm{sec}} - \sum_{p=1}^{N_g}\sum_{q=1}^{N_g^{\mathrm{act}}}\sum_{\ell=1}^{N_g} w_p w_q w_\ell W_{pq}W_{q\ell}\boldsymbol{b}_p^\sigma\boldsymbol{b}_\ell^{\sigma T}. \tag{90}$$

It can be recognized that the consistent tangent stiffness matrix obtained in Eq. (90) is symmetric. The symmetry comes from a correct definition of an associative rule for nonlocal dissipative phenomenon, namely the thermodynamically justified framework constructed in Sections 3 and 4 plays a fundamental role for the symmetry of the tangent stiffness matrix as well as for the symmetry of the continuum tangent moduli tensor found in Eq. (52).

8. Numerical results

The constitutive model previously developed and the overall nonlocal incremental finite element procedure presented in the last Section has been implemented in the research oriented FE program FEAP written by R.L. Taylor (see Zienkiewicz and Taylor, 2000). Two finite elements have been implemented. The first is a one-dimensional bar and the second is a four node plain stress two-dimensional finite element. In the following subsections the numerical applications related to one- and two-dimensional applications are discussed.

8.1. One-dimensional bar in tension

The first numerical application regards a one-dimensional bar in simple uniform traction. The bar, depicted in Fig. 2, is of length $L = 100$ mm and of unitary cross section area. The internal energy chosen for the present application is the one recently proposed by Comi and Perego (2001c) for plain concrete

$$\psi_{\text{in}}(\xi) = K(1-\xi)\sum_{j=1}^{N}\frac{N!}{j!}\ln^{j}\left(\frac{c^*}{1-\xi}\right), \tag{91}$$

where K, c^* and N are material parameters and, by definition, $0! = 1$. Fig. 3 shows the local stress–strain relation in traction obtained considering that $\chi = \partial\psi_{\text{in}}/\partial\xi = K\ln^{N}(c^*/(1-\xi))$ and using the local damage activation condition $Y - \chi = 0$. As shown in Fig. 3, the local model is such that, in one-dimension, the stress vanishes only asymptotically, for $\varepsilon \to \infty$, but with a bounded fracture energy density.

The numerical simulation of the uniaxial tensile test has been performed by one dimensional finite elements with constant strain and linear displacement interpolation functions. The analysis has been carried out under displacement control and an arch-length method has been used in order to control possible snap-back branch in the structural response of the bar. The material parameters K, c^* and N have been chosen such to reproduce the properties of the concrete employed in a 2-D traction experiment test performed by Hassanzadeh (1991), hence assuming the Young modulus $E = 36\,000$ MPa, the tensile strength

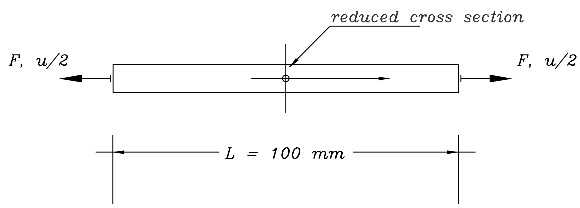

Fig. 2. Sketch of one-dimensional bar in simple tension.

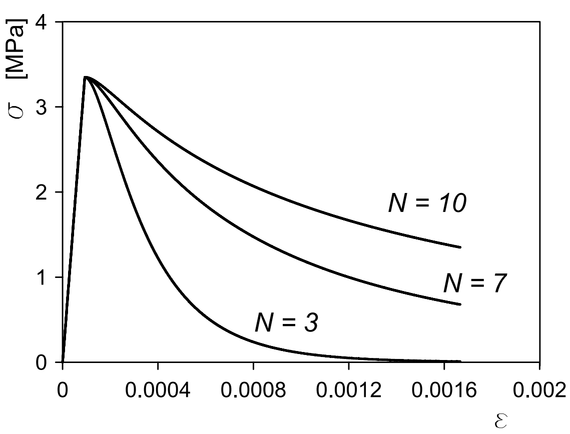

Fig. 3. One-dimensional stress–strain curve obtained by the model of Comi and Perego (2001a,b,c) with different values of the coefficient N.

$\sigma_f = 3.35$ MPa, the appropriate parameter values are $K = 1.605632 \times 10^{-11}$ MPa, $c^* = 148.413159$, and $N = 10$. Moreover, in order to trigger the damage localization, the cross section area of the central element has been reduced of 0.001%.

The attenuation function employed in the analysis is the Gauss function $e^{-\pi(r/\ell_0)^2}$ and the internal length $\ell_0 = 4.5$ mm. It is of interest to observe the different role played by N and by the internal length ℓ_0. Namely, N influences the fragility of the overall structural response, that is, the fragility increases when N decreases, while the internal length ℓ_0 affects the width of the damage localized zone.

The force–displacement response of the bar is reported in Fig. 4, and it can be observed that the solution correctly converges as the number of elements is uniformly increased. Fig. 5 depicts the evolution of the damage distribution inside the bar as the imposed displacements at the boundaries increase.

8.2. 2-D plate in tension

The second numerical application regards a direct traction test of a four notched concrete sample of which experimental tests have been performed by Hassanzadeh (1991). Fig. 6 shows the geometry and the loading condition of the sample. The numerical analysis has been carried out under the hypothesis of plane stress condition and using the same constitutive model and material parameters presented in the previous Section 8.1.

The analysis has been performed for the four different meshes shown in Fig. 7. The first two discretizations, mesh A and B, are regular meshes with 222 and 968 elements respectively; the last two meshes, mesh C and D have been constructed realizing a finer discretization in the central zone with 864 and 2320 elements respectively. Fig. 8 shows a surface plot reporting the ratio $\Omega_r(\boldsymbol{x})/\Omega_\infty$ relative to the data of mesh B. It can be observed that for interior points it is equal to unity (with some numerical approximation in the zones where the mesh changes density) whereas it decreases as the points approaches the body's boundary.

In order to compare numerical and experimental results the force–displacement curves have been evaluated, looking for the resultant reaction, R, at the base where the displacement is imposed and the relevant notch opening. In Fig. 9 the force–displacement curves, obtained for the four meshes, are reported and compared to the experimental curve of Hassanzadeh (1991). It can be observed a good agreement between the numerical and experimental data and a very satisfactory mesh independent result.

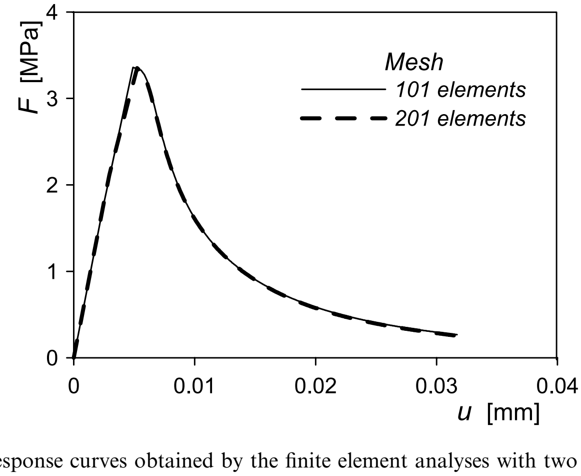

Fig. 4. Force–displacement response curves obtained by the finite element analyses with two different discretization meshes.

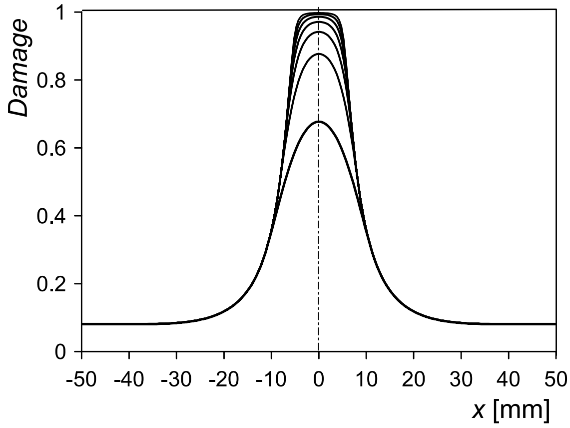

Fig. 5. Damage localization profiles in the bar for increasing assigned displacement at the boundaries.

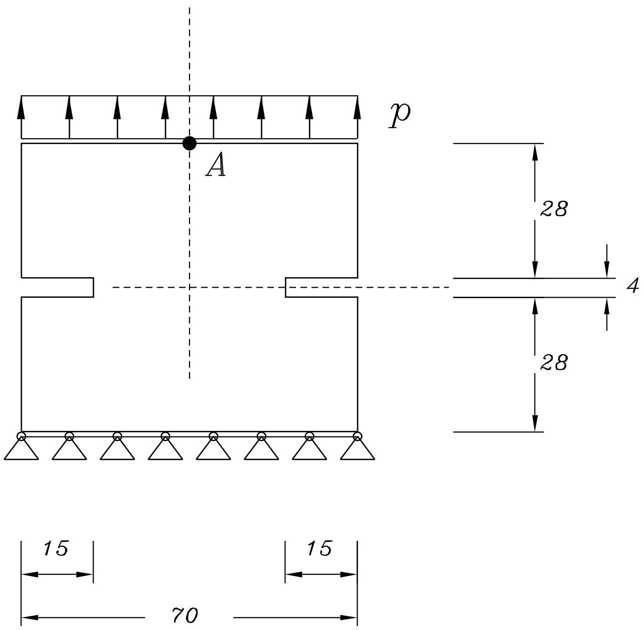

Fig. 6. Two-dimensional concrete specimen subjected to direct tension test. Geometry and load condition.

The numerical analysis has also shown that the damage starts at the notches, propagates to the center zone of the specimen and finally it extends to all the central zone up to a full formation of a damage band. Fig. 10 shows the regularized damage contours at two different load levels. Namely, Fig. 10a shows the damage distribution for the loading level near the peak; Fig. 10b shows the damage distribution in the structural softening branch and it can be observed that the localized damage distribution is fully developed. The two conditions correspond respectively to the points (a) and (b) reported in Fig. 9.

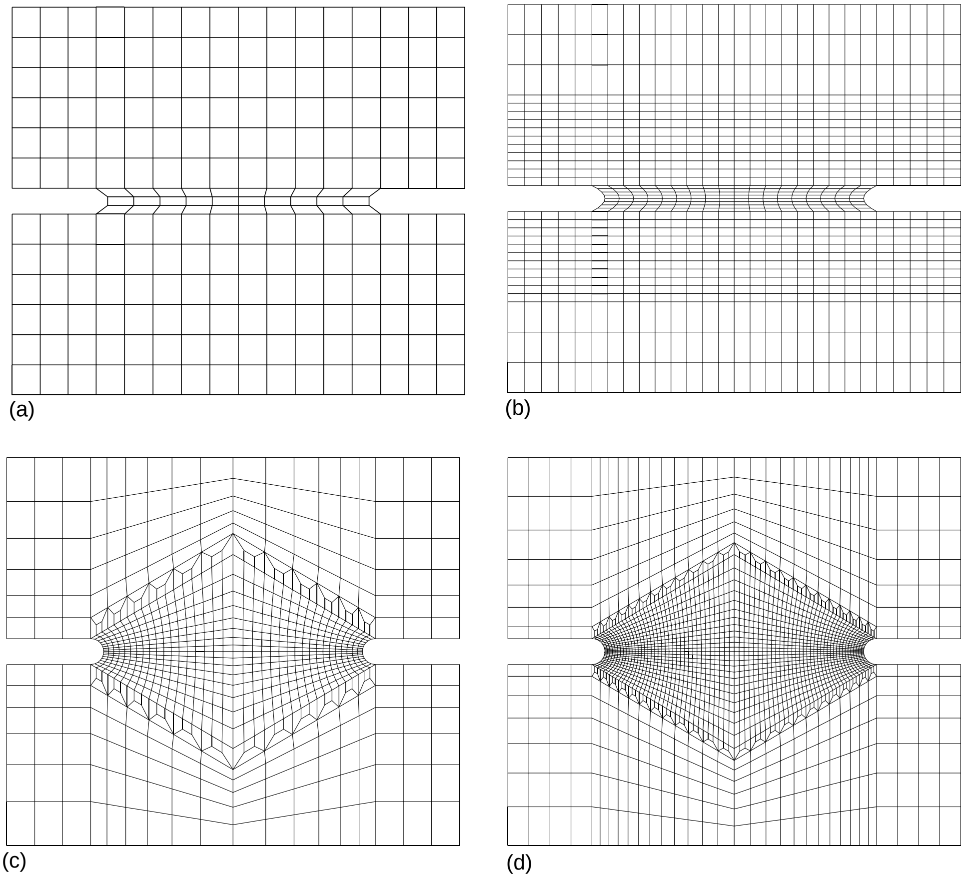

Fig. 7. Four different meshes adopted in the finite element simulations: Meshes: (a) 222 elements, (b) 968 elements, (c) 864 elements, (d) 2320 elements.

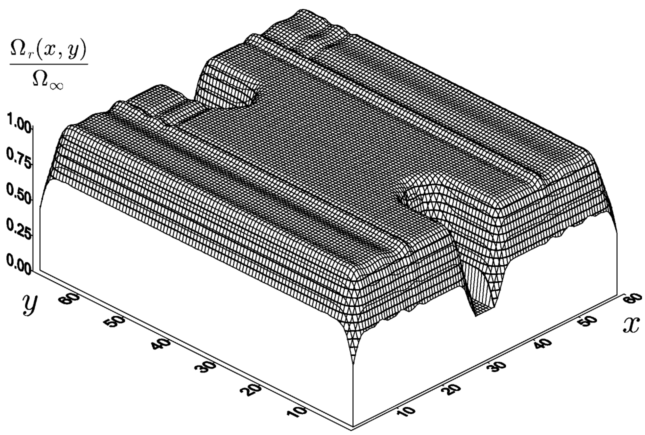

Fig. 8. Surface plot showing the distribution of $\Omega_r(x)/\Omega_\infty$ for the structure of Fig. 6 discretized with the mesh B.

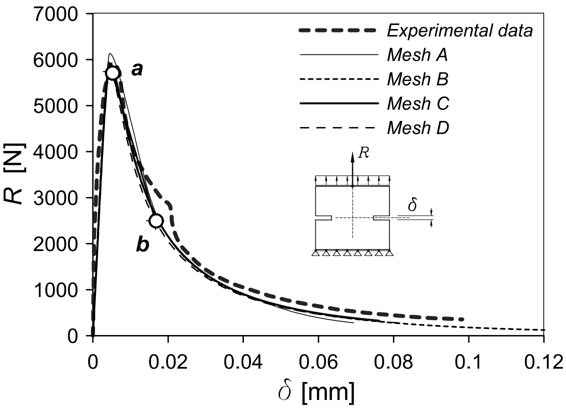

Fig. 9. Diagram showing traction reaction R versus the open displacement at the notch for the four meshes of Fig. 7. The experimental results obtained by Hassanzadeh (1991) are also reported for comparison.

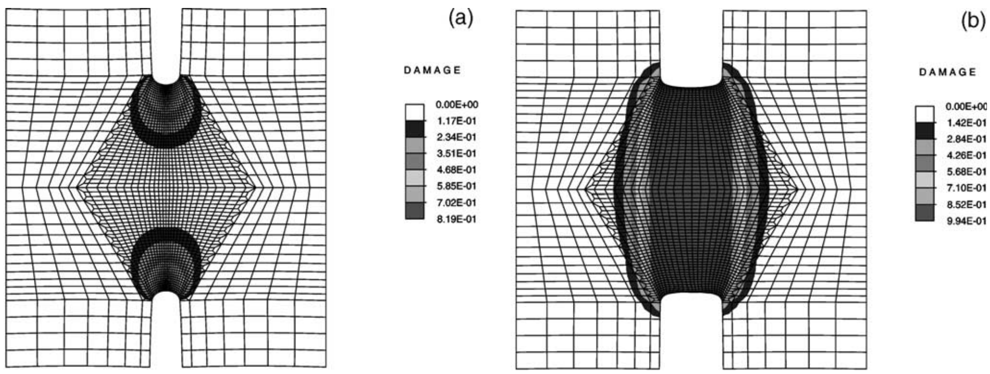

Fig. 10. Damage contour maps related to the solution obtained with the mesh D. (a) Damage distribution near the peak of the force–displacement curve (point (a) of Fig. 9). (b) Damage distribution in the softening branch of the force–displacement curve (point (b) of Fig. 9).

9. Summary and conclusions

A new formulation for nonlocal damage models has been presented. This formulation closely follows a recent thermodynamic consistent approach that requires the definition of a damage nonlocal variable in conjunction with a nonlocal measure of the energy release rate. In order to preserve uniform damage fields, as well as uniform energy release rate fields, a nouvelle definition of the nonlocal integral operator has been presented. At difference with the most common approach, that makes use of a spatial weight function that changes its shape as the considered material point approaches the boundaries of the body, the present formulation redefines the nonlocal integral operator as the sum of a variable local contribution plus a standard nonlocal term with fixed space weight function. The difference between the present new approach and the traditional one becomes apparent only getting close to the boundaries, whereas far from the boundaries, or in case of unbounded media, it is shown that the local contribution vanishes and then this formulation is fully equivalent to the classical one.

The definition of self-adjoint nonlocal integral operator, in conjunction with the overall nonlocal thermodynamic consistency and the damage associative rules, has permitted the derivation of a nonlocal damage theory that enjoys many valuable properties as, for instance, the symmetry of the tangent operators and the existence of variational principles for the material and the structural rate response characterization.

The implementation of the model in a standard finite element program has been also presented, showing the capacity of the model to reproduce physically meaningful and mesh objective numerical results.

The specific damage constitutive model presented in the numerical applications is quite simple and it is intended only for the discussion of the new general nonlocal damage framework presented. In a forthcoming paper the issue of a damage model able to describe the unilateral tension-compression behaviour, the mixed shear damage modes and also the production of permanent strains of plastic nature will be addressed.

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Appendix A. One-dimensional localization analysis

In this appendix the capacity of the nonlocal damage model to re-establish a well-posed problem is investigated analyzing the propagation speed of stress acceleration waves. Namely it is necessary to ensure that such velocity does not become imaginary.

Following the approach of Larsy and Belytschko (1988) and Pijaudier-Cabot and Benallal (1993), and also considering a recent contribution of Comi and Rizzi (2000), the study of the propagation of acceleration waves is here considered for a one-dimensional bar of infinite length. Considering the bar in an initial uniform state of strain and damage ($\bar{d}(x) = d(x) = \text{const.}\ \varepsilon(x) = \text{const.}$) the stress–strain relation of Eq. (49) written for one-dimensional problems reads

$$\dot{\sigma} = (1-d)E\frac{\partial \dot{u}}{\partial x} - E\varepsilon\dot{\bar{d}}. \tag{A.1}$$

The equation of motion in rate form linearized about the uniform initial state is

$$\frac{\partial \dot{\sigma}}{\partial x} = \rho\frac{\partial^2 \dot{u}}{\partial t^2}. \tag{A.2}$$

Substituting Eq. (A.1) in Eq. (A.2) and considering the explicit form of the integral regularization operator for an infinite length bar, the following relation is obtained

$$(1-d)E\frac{\partial^2 \dot{u}}{\partial x^2} - E\varepsilon\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\alpha(x-y)\dot{d}(y)\,\mathrm{d}y - \rho\frac{\partial^2 \dot{u}}{\partial t^2} = 0. \tag{A.3}$$

Let us now consider a linear comparison one-dimensional solid where the linearization is realized under the assumption of damage loading and in the initial condition of uniform damage and uniform strain field as before. The damage loading consistency condition requires

$$\dot{\phi} = \dot{\bar{Y}} - \dot{\chi} = \int_{-\infty}^{\infty}\alpha(x-y)\varepsilon(y)E\left.\frac{\partial \dot{u}}{\partial x}\right|_{y}\mathrm{d}y - h(\xi)\dot{d} = 0, \tag{A.4}$$

which considering the uniform initial strain state can be rewritten as

$$\dot{\phi} = \dot{\bar{Y}} - \dot{\chi} = E\varepsilon \int_{-\infty}^{\infty} \alpha(x-y) \frac{\partial \dot{u}}{\partial x}\bigg|_y \mathrm{d}y - h(\xi)\dot{d} = 0. \tag{A.5}$$

Let us consider for the equation of motion, Eq. (A.3), and for the damage consistency condition, Eq. (A.5), harmonic wave solutions of the following form

$$\dot{u}(x,t) = \dot{u}_0 \mathrm{e}^{\mathrm{i}q(x+ct)}, \quad \dot{d}(x,t) = \dot{d}_0 \mathrm{e}^{\mathrm{i}q(x+ct)}, \tag{A.6}$$

where q is a wave number and c is the speed of propagation. Substituting Eqs. (A.6), and their partial derivatives with respect to x and t, into Eqs. (A.3) and (A.5), the following relations are obtained

$$\{(\rho q^2 c^2 - (1-d)Eq^2)\dot{u}_0 - (\mathrm{i}qA(q)E\varepsilon)\dot{d}_0\}\mathrm{e}^{\mathrm{i}q(x+ct)} = 0, \tag{A.7}$$

$$\{(\mathrm{i}qA(q)E\varepsilon)\dot{u}_0 - h(\xi)\dot{d}_0\}\mathrm{e}^{\mathrm{i}q(x+ct)} = 0, \tag{A.8}$$

where $A(q)$ is the Fourier transform of $\alpha(r)$, namely $A(q) = \int_{-\infty}^{\infty} \alpha(r)\mathrm{e}^{-\mathrm{i}qr}\mathrm{d}r$. Eqs. (A.7) and (A.8) must be verified for every instant t and at every point x. Then, Eqs. (A.7) and (A.8) can be viewed as an homogeneous linear system of two equations, that in order to have a solution distinct from the trivial one, must have the determinant of the coefficient matrix equal to zero, namely

$$c^2\rho h(\xi) - (1-d)h(\xi)E + A^2(q)E^2\varepsilon^2 = 0, \tag{A.9}$$

which solved with respect to the speed c gives

$$c = c_e\sqrt{1 - \frac{E\varepsilon^2 A^2(q)}{(1-d)h(\xi)}}, \tag{A.10}$$

where $c_e = ((1-d)E/\rho)^{1/2}$ is the elastic speed in case of unloading. Eq. (A.10) shows the dispersive nature of the nonlocal damage media. In fact, the speed of propagation depends on the wave number q, which implies that, if a general shape stress wave travels along the bar, the wave shape will change since each harmonic component would travel at a different speed.

It is of interest to find a wave number q_{cr} which makes the propagation speed vanishing, i.e. $c = 0$. This specific value is obtained solving the equation

$$A^2(q_{\mathrm{cr}}) = \frac{(1-d)h(\xi)}{E\varepsilon^2}. \tag{A.11}$$

For instance in case of Gauss attenuation function $\alpha(r) = \mathrm{e}^{-\pi r^2/\ell_0^2}$, which gives as Fourier transform $A(q) = \mathrm{e}^{-q^2\ell_0^2/4\pi}$, the critical wave number is

$$q_{\mathrm{cr}} = \frac{\sqrt{2\pi}}{\ell_0}\left[\ln\left(\frac{E\varepsilon^2}{(1-d)h(\xi)}\right)\right]^{1/2}. \tag{A.12}$$

A better understanding of the localization analysis is achieved by defining the related *critical wave length*, $\Lambda_{\mathrm{cr}} = 2\pi/q_{\mathrm{cr}}$, that with reference to Eq. (A.12) gives

$$\Lambda_{\mathrm{cr}} = \sqrt{2\pi}\ell_0\left[\ln\left(\frac{(1-d)h(\xi)}{E\varepsilon^2}\right)\right]^{1/2}. \tag{A.13}$$

It is worth noting that waves with wave length $\Lambda < \Lambda_{\mathrm{cr}}$ do not propagate in the bar. Then, Λ_{cr} can be considered a measure of the width of the static damage active localization band. For instance, considering the internal energy of Comi and Perego (2001c) given in Eq. (91), for which (choosing the coefficients K, n,

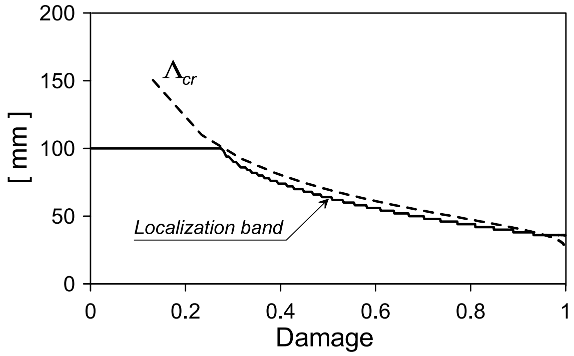

Fig. 11. Comparison between the critical wave length obtained by the bifurcation analysis (Eq. (A.13)) and the width of the active damage band obtained by the numerical simulation described in Section 8.1.

c^* as in the application of the Section 8.1), $h(\xi) = Kn/(1-d)\ln^{n-1}(c^*/(1-d))$, Fig. 11 shows the evolution of Λ_{cr} as function of the initial damage state obtained by Eq. (A.13) compared to the width of the damage active zone obtained checking numerically the damage active Gauss points in the finite element analysis. A very good agreement can be observed.

References

Addessi, D., Marfias, S., Sacco, E., 2002. A plastic nonlocal damage model. Comp. Meth. Appl. Mech. Eng. 191, 1291–1310.

Aifantis, E.C., 1984. On the microstructural origin of certain inelastic models. J. Eng. Mater. Technol. 106, 326–330.

Aifantis, E.C., 1992. On the role of gradients in the localization of deformation and fracture. Int. J. Eng. Svi. 30, 1279–1299.

Bažant, Z.P., 1984. Nonlocal damage theory based on micromechanics of crack interactions. J. Eng. Mech. 120, 1512–1533.

Bažant, Z.P., 1991. Why continuum damage is nonlocal: micromechanics arguments. J. Eng. Mech. 1117, 1070–1087.

Bažant, Z.P., Cedolin, L., 1979. Blunt crack band propagation in finite element analysis. J. Eng. Mech. ASCE 120, 297–315.

Bažant, Z.P., Oh, B.-H., 1983. Crack band theory for fracture of concrete. Mater. Struct. (RILEM) 16, 155–177.

Bažant, Z.P., Planas, J., 1998. Fracture and Size Effect in Concrete and Other Quasibrittle Materials. CRC Press, Boce Raton and London.

Benvenuti, E., Borino, G., Tralli, A., 2000. A thermodynamically consistent nonlocal formulation for elasto-damaging materials: theory and computations. In: Oñate, E. (Ed.), CD Proc. ECCOMAS 2000. CIMNE, Barcelona, Spain.

Benvenuti, E., Borino, G., Tralli, A., 2002. A thermodynamically consistent nonlocal formulation for damaging materials. Eur. J. Mech. A/Solids 21 (4), 535–553.

Borino, G., Failla, B., 2000. Thermodynamically consistent plasticity models with local and nonlocal internal variables. In: Oñate, E. (Ed.), CD Proc. ECCOMAS 2000. CIMNE, Barcelona, Spain.

Borino, G., Fuschi, P., Polizzotto, C., 1999. A thermodynamic approach to nonlocal plasticity and related variational principles. ASME J. Appl. Mech. 66, 952–963.

Comi, C., 2001. A nonlocal model with tension and compression damage mechanisms. Eur. J. Mech. A/Solids 12, 1–22.

Comi, C., Perego, U., 2001a. Fracture energy based bi-dissipative damage model for concrete. Int. J. Solid Struct. 38, 6427–6454.

Comi, C., Perego, U., 2001b. Symmetric and non-symmetric non-local damage formulations: an assessment of merits. In: CD Proc. ECCM 2001. Cracow, Poland, 2001.

Comi, C., Perego, U., 2001c. Numerical aspects of nonlocal damage analyses. Eur. J. Finite Elements 10, 227–242.

Comi, C., Rizzi, E., 2000. On bifurcation in local and nonlocal materials with tension and compression damage. In: CD Proc. ECCOMAS 2000. Barcelona, Spain.

de Borst, R., Mühlhaus, H.-B., 1992. Gradient dependent plasticity: formulation and algorithmic aspects. Int. J. Num. Meth. Eng. 35, 521–539.

de Borst, R., Pamin, J., 1996. Gradient plasticity in numerical simulation of concrete cracking. Eur. J. Mech. A/Solid 15, 295–320.

Edelen, D.G.B., Laws, N., 1971. On the thermodynamics of systems with nonlocality. Arch. Rat. Mech. Anal. 43, 24–35.

Edelen, D.G.B., Green, A.E., Laws, N., 1971. Nonlocal continuum mechanics. Arch. Rat. Mech. Anal. 43, 36–44.

Eringen, A.C., 1981. On nonlocal plasticity. Int. J. Eng. Sci. 19, 1461–1474.

Este, G., Willam, K., 1992. Fracture energy formulation for inelastic behaviour of plain concrete. J. Eng. Mech. ASCE 120, 1983–2011.
Ganghoffer, J.-F., Sluys, L.J., de Borst, R., 1999. A reappraisal of nonlocal mechanics. Eur. J. Mech. A/Solid 18, 17–46.
Hassanzadeh, M., 1991. Behaviour of fracture process zones in concrete influenced by simultaneously applied normal and shear displacements, Ph.D. Thesis Lund Institute of Technology, Lund.
Jirásek, M., 1998. Nonlocal models for damage and fracture: comparison of approaches. Int. J. Solid Struct. 35, 4133–4145.
Jirásek, M., 2001. Computational aspects of nonlocal models. In: CD Proc. of ECCM 2001. Cracow, Poland, 2001.
Jirásek, M., Rolshoven, S., in press. Comparison of integral-type non-local plasticity models for strain-softening materials. Int. J. Eng. Science.
Krajcinovic, D., 1996. Damage Mechanics. North-Holland, Amsterdam.
Larsy, D., Belytschko, T., 1988. Localization limiters in transient problems. Int. J. Solid Struct. 24, 581–597.
Lemaitre, J., Chaboche, J.-L., 1990. Mechanics of Solids Materials. University Press, Cambridge.
Luciano, R., Willis, J.R., 2001. Non-local constitutive response of a random laminate subjected to configuration-dependent body force. J. Mech. Phys. Solid 49, 431–444.
Noll, W., 1972. A new mathematical theory of simple materials. Arch. Rational Mech. Anal. 48, 1–50.
Patzák, B., Jirásek, M., 2001. Consistent tangent stiffness for nonlocal materilal models. In: CD Proc. ECCM 2001. Cracow, Poland.
Pijaudier-Cabot, J., Benallal, A., 1993. Strain localization and bifurcation in a nonlocal continuum. Int. J. Solid Struct. 30 (13), 1761–1775.
Pijaudier-Cabot, G., Bažant, Z.P., 1987. Nonlocal damage theory. J. Eng. Mech. 113, 1512–1533.
Polizzotto, C., 2001. Nonlocal elasticity and related variational principles. Int. J. Solid Struct. 38, 7359–7380.
Polizzotto, C., 2002. Personal communications.
Polizzotto, C., Borino, G., Fuschi, P., 1998. A thermodynamic consistent formulation of nonlocal and gradient plasticity. Mech. Res. Commun. 25 (1), 75–82.
Simo, J.C., Hughes, T.J.R., 1998. Computational Inelasticity. Springer-Verlag, Berlin.
Strömberg, L., Ristimnaa, M., 1996. FE-formulation of a nonlocal plasticity theory. Comput. Meth. Appl. Mech. Engrg. 136, 127–144.
Zienkiewicz, O.C., Taylor, R.L., 2000. The finite element method. Butterworth-Heinemann, Oxford.